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The equivalent elastic crack procedure as applied to internally pressurised crack problems

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Analysis of a specific idealised model demonstrates the limitations of the equivalent elastic crack procedure when it is applied to internally pressurised crack problems.

1. Introduction

The cohesive zone description has been extensively used to study the failure of quasi-brittle materials, such as concrete, rock, ice, tough ceramics and some composites. With this description, when it is used for two-dimensional crack situations, an infinitesimally thin cohesive zone forms at a crack tip, the zone being characterised by a material specific relation between the tensile stress (p) across the zone, and the relative displacement (v) between the zone faces. $p(v)$ decreases as v increases, the cohesive zone thus exhibiting softening characteristics. The maximum stress that the zone is able to withstand is p_c and the maximum is associated with the leading edge of the cohesive zone. The zone is said to be fully developed when the stress p falls to zero at the trailing edge of the zone, a situation that is assumed to be attained when the displacement v attains a critical value v_c. For a positive loading situation where the crack tip stress intensity increases as a crack extends, for a fixed applied loading, and with a general p-v cohesive zone softening behaviour, the maximum stress, and thereby failure, is attained prior to the cohesive zone's full development.

The cohesive zone description is frequently used in conjunction with the effective elastic crack procedure. With this procedure, it is recognised that the far-field behaviour for a cohesive zone model coincides with the behaviour for the corresponding effective elastic crack model, when the cohesive zone size is small compared with the geometrical dimensions of the configuration under consideration; the effective elastic crack then has its tip somewhere within the actual cohesive zone. The J contour integral is a far-field parameter, though with the special property that it is path independent for any path completely surrounding a cohesive zone, and this property can be exploited [1] so as to give the first order deviation from the LEFM criterion for crack extension. This deviation relation, which is strictly valid only when the cohesive zone size is small compared with a configuration's characteristic dimensions, has been used more generally by Bazant and co-workers [2] as an approximation for situations where the zone size is not necessarily small compared with these dimensions.

It is against this general background that the present paper examines the behaviour of a crack which is subjected to an internal pressure, a state of affairs that is relevant to the problem of hydraulic fracturing of rock. In such a situation, evaluation of the J integral around a path that spreads beyond the cohesive zone will include a contribution from the pressurized crack surface, and the J integral will not then be path independent. Consequently, it would appear unlikely that the equivalent elastic crack procedure can be applied to internally pressurised crack problems; this paper analyses a specific idealised model whose results do indeed demonstrate the limitation of the procedure when it is applied to such situations.

2. General theoretical background

Consider a solid in which there is a crack of initial depth a_0, D is a characteristic geometrical dimension and σ_N is a nominal stress. The stress intensity defined with regard to a crack of depth a can be expressed in the general form

$$K_I = \sigma_N \sqrt{D} S(a/D) \quad (1)$$

where $S(a/D)$ is a geometrical shape factor. Following Planas and Elices [1] and using the equivalent elastic crack procedure, Rice's J integral [3], which is a far-field parameter, can be written in the form

$$J = \frac{1}{E_0}[K_I(a_0 + \Delta a_E)]^2 \quad (2)$$

where $E_0 = E/(1-\nu^2)$ for plane strain deformation, with E being Young's modulus and ν being Poisson's ratio, while Δa_E is the elastically equivalent size of cohesive zone, which need not necessarily be fully developed. A two term expansion of relation (2) gives

$$J = \frac{K_{IN}^2}{E_0}\left[1 + \frac{2S_0'}{S_0} \cdot \frac{\Delta a_E}{D}\right] \quad (3)$$

where S_0 and $S_0' = dS/da$ are defined with respect to $a = a_0$; K_{IN} is the stress intensity defined with regard to the initial crack tip, i.e. it is defined by relation (1) but with $a = a_0$. Now J, as given by relation (3), can be equated with the area W_F under the p-v softening curve up to a value of v that is equal to the displacement v_T at the trailing edge of the cohesive zone, i.e. at the initial crack tip. Thus,

$$\frac{K_{IN}^2}{E_0}\left[1 + \frac{2S_0'}{S_0}\cdot\frac{\Delta a_E}{D}\right] = W_F = \int_0^{v_T} p(v)\,dv \qquad (4)$$

With positive geometries, i.e. those for which the shape factor $S(a/D)$ increases with crack extension, general arguments have been used [1] to show that, although the maximum stress is attained before the zone is fully developed, the maximum stress is given, for the large D situation, by relation (4), but with $\Delta a_E \equiv c_f$ and $w_F \equiv G_F$. c_f is the elastically equivalent size of the fully developed cohesive zone associated with the extreme case of a semi-infinite crack in a remotely loaded infinite solid and G_F is the specific fracture energy appropriate for the p-v softening law and is given by the expression

$$G_F = \int_0^{v_c} p(v)\,dv \qquad (5)$$

Consequently, the resulting expression giving the maximum stress is

$$\frac{E_0 G_F}{K_{IN,MAX}^2} = \frac{K_{IC}^2}{K_{IN,MAX}^2} = 1 + \frac{2S_0'}{S_0}\cdot\frac{c_f}{D} \qquad (6)$$

with $K_{IC} = [E_0 G_F]^{1/2}$ being the fracture toughness of the material. Expression (6), which reduces to the LEFM criterion $K_{IN,MAX} = K_{IC}$ for the limiting case where $c_f/D \to 0$, is valid for small c_f/D, i.e. for large structural dimensions. However, as mentioned in Section 1, expression (6) has been used by Bazant and co-workers [2] as an approximate representation of the true state of affairs for a wide range of c_f/D values.

In the next section, we explore the viability of the equivalent elastic crack procedure, as manifested by relation (6), with regard to internally pressurised crack problems.

3. Analysis of a specific idealised model

To simplify the considerations, it is assumed that the stress within the cohesive zone remains constant at the value p_c until the displacement v attains the critical value v_c when the stress is assumed to fall abruptly from p_c to zero. This is the classic Dugdale-Bilby-Cottrell-Swinden (DBCS) representation [4, 5], and with this specific behaviour, the attainment of maximum stress is associated with the full development of the cohesive zone.

The particular model that will be analysed in this section is that of a two-dimensional Mode I crack of length $2a_0$ in an infinite solid that is subjected to an applied tensile stress σ_A while the crack is also subjected to an internal pressure σ_I; there are cohesive zones at the crack tip, and the tensile stress within these zones is p_c. Now for the case where $\sigma_I = 0$, the applied tensile stress σ_A required to cause crack extension, i.e. to induce a displacement v_c at a crack tip, is given by the expression [5]

$$\frac{\sigma_A}{p_c} = \frac{2}{\pi}\sec^{-1}\exp\left\{\frac{\pi E_0 v_c}{8 p_c a_0}\right\} \qquad (7)$$

With regard to this relation, it should be noted that σ_A and $(\sigma_A - p_c)$ are the stresses acting upon the dislocations that represent the relative displacement across respectively the crackfaces and the cohesive zones. For the case where we have a combination of external stress σ_A and internal pressure σ_I, the corresponding stresses are $(\sigma_A + \sigma_I)$ and $(\sigma_A - p_c)$. Thus we can obtain the expression giving the σ_A/σ_I combination of stresses required to cause crack extension by replacing σ_A and p_c in relation (7) by respectively $(\sigma_A + \sigma_I)$ and $(p_c + \sigma_I)$, i.e.

$$\frac{(\sigma_A + \sigma_I)}{(p_c + \sigma_I)} = \frac{2}{\pi}\sec^{-1}\exp\left\{\frac{\pi E_0 v_c}{8(p_c + \sigma_I)a_0}\right\} \qquad (8)$$

Now with the DBCS representation, the specific fracture energy $G_F = p_c v_c$ whereupon $K_{IC} = [E_0 p_c v_c]^{1/2}$ and thus with $\theta = \pi K_{IC}^2/8p_c^2 a_0$, $x = (\sigma_A + \sigma_I)/p_c$ and $\lambda = \sigma_I/(\sigma_I + \sigma_A)$ it being assumed that we have proportional loading, then relation (8) can be written in the form

$$\cos\left\{\frac{\pi x}{2(1+\lambda x)}\right\} = \exp\left\{\frac{-\theta}{(1+\lambda x)}\right\} \qquad (9)$$

Since we are concerned with the situation where θ and x are both small, and noting that λ ranges between 0 and 1, expansion of both sides of relation (9) to the first three terms gives

$$1 - \frac{\pi^2 x^2}{8(1+\lambda x)^2} + \frac{\pi^4 x^4}{384(1+\lambda x)^4} = 1 - \frac{\theta}{(1+\lambda x)} + \frac{\theta^2}{2(1+\lambda x)^2} \qquad (10)$$

whereupon

$$x^2 = \frac{8\theta(1+\lambda x)}{\pi^2} + \frac{\pi^2 x^4}{48(1+\lambda x)^2} - \frac{4\theta^2}{\pi^2} \qquad (11)$$

Relation (11) shows that θ can be expressed in the form

$$\frac{8\theta}{\pi^2} = x^2 - \lambda x^3 + \left(\lambda^2 + \frac{\pi^2}{24}\right)x^4 \qquad (12)$$

to the first three terms in increasing powers of x, whereupon we can express x in terms of θ as follows:

$$x = \frac{8^{1/2}\theta^{1/2}}{\pi} + \frac{4\lambda\theta}{\pi^2} + \frac{8^{1/2}}{2\pi^3}\left[2\lambda^2 - \frac{\pi^2}{3}\right]\theta^{3/2} \qquad (13)$$

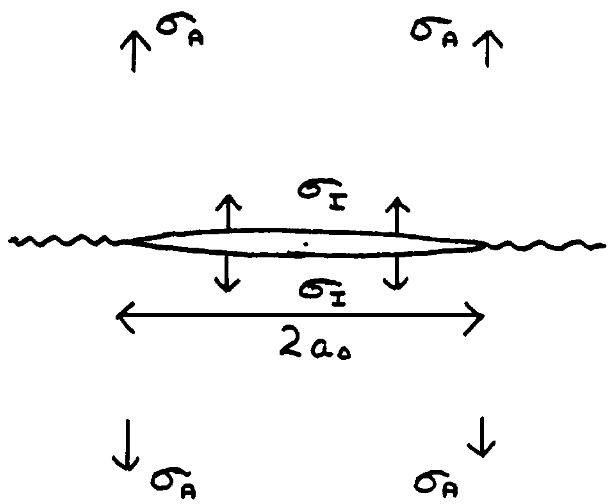

Figure 1 The cohesive zone model analysed in Section 3.

to the first three terms in increasing powers of θ. It then follows that

$$\frac{1}{x^2} = \frac{\pi^2}{8\theta}\left[1 - \frac{\lambda 8^{1/2}\theta^{1/2}}{\pi} + \theta\left(\frac{1}{3} + \frac{5\lambda^2}{2\pi^2}\right)\right] \quad (14)$$

again to the first three terms in increasing powers of θ. Thus with $x = (\sigma_A + \sigma_I)/p_c$, $\theta = \pi K_{IC}^2/8p_c^2 a_0$ and $K_{IN,MAX} = (\sigma_A + \sigma_I)(\pi a_0)^{1/2}$, this relation can be rewritten in the form

$$\frac{K_{IC}^2}{K_{IN,MAX}^2} = 1 - \frac{\lambda}{\pi^{1/2}} \cdot \frac{K_{IC}}{p_c a_0^{1/2}} + \frac{\pi K_{IC}^2}{24 p_c^2 a_0}\left(1 + \frac{15\lambda^2}{2\pi^2}\right) \quad (15)$$

with $\lambda = \sigma_I/(\sigma_I + \sigma_A)$. On the other hand, the crack extension condition as given by the equivalent elastic crack procedure [see relation (6) with the function $S \equiv (\pi a/D)^{1/2}$)] is

$$\frac{K_{IC}^2}{K_{IN,MAX}^2} = 1 + \frac{c_f}{a_0} = 1 + \frac{\pi K_{IC}^2}{24 p_c^2 a_0} \quad (16)$$

since $c_f = \pi K_{IC}^2/24 p_c^2$ with the DBCS cohesive zone representation [6].

Comparison of relations (15) and (16) shows that when there is no internal pressure, i.e. $\sigma_I = 0$ or $\lambda = 0$, then the equivalent elastic crack procedure gives, not surprisingly, the same crack extension criterion as does an exact analysis based on the DBCS representation. However, if the crack is internally pressurised, i.e. $\sigma_I \neq 0$ or $\lambda \neq 0$, then the procedure clearly does not give the correct crack extension condition. Indeed the first order deviation from the LEFM criterion involves a term $K_{IC}/p_c a_0^{1/2}$ (see relation (15)) which is not the case when there is no internal pressure.

4. Concluding comments

The analysis of the specific idealised DBCS cohesive zone model in the preceding section has confirmed the supposition in the Introduction that the equivalent elastic crack procedure cannot be used for internally pressurised crack problems. The reason for this is that the procedure depends on the principle that the far-field behaviour for a cohesive zone model coincides with the behaviour for the corresponding effective elastic crack model, when the cohesive zone size is small compared with the geometrical dimensions of the configuration under consideration. The J contour integral is a far-field parameter, though with the special property that it is path independent for any path completely surrounding a cohesive zone, and this property can be exploited so as to give the deviation from the LEFM crack extension condition. This procedure works satisfactorily as regards an externally applied stress. However, as demonstrated in the preceding section, it does not work with an internally pressurised crack problem. In such a situation, evaluation of the J integral around a path that spreads beyond the cohesive zone will include a contribution from the pressurized crack surface, and the J integral will not then be path independent.

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